Network Analysis:

The Hidden Structures behind the Webs We Weave

17-213 / 17-668

Communities

Thursday, September 28, 2023

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2-min Quiz, on Canvas

Quick Recap – Last Tuesday’s Lecture

Carefully examine assumptions of network measures:

- Centrality metrics do not accurately predict “power” in negatively connected exchange networks (zero-sum)

Make the hidden assumptions explicit:

- Bonacich power centrality explicates the assumption with the beta parameter
- Better prediction of power use in experimental data (higher resource gains)

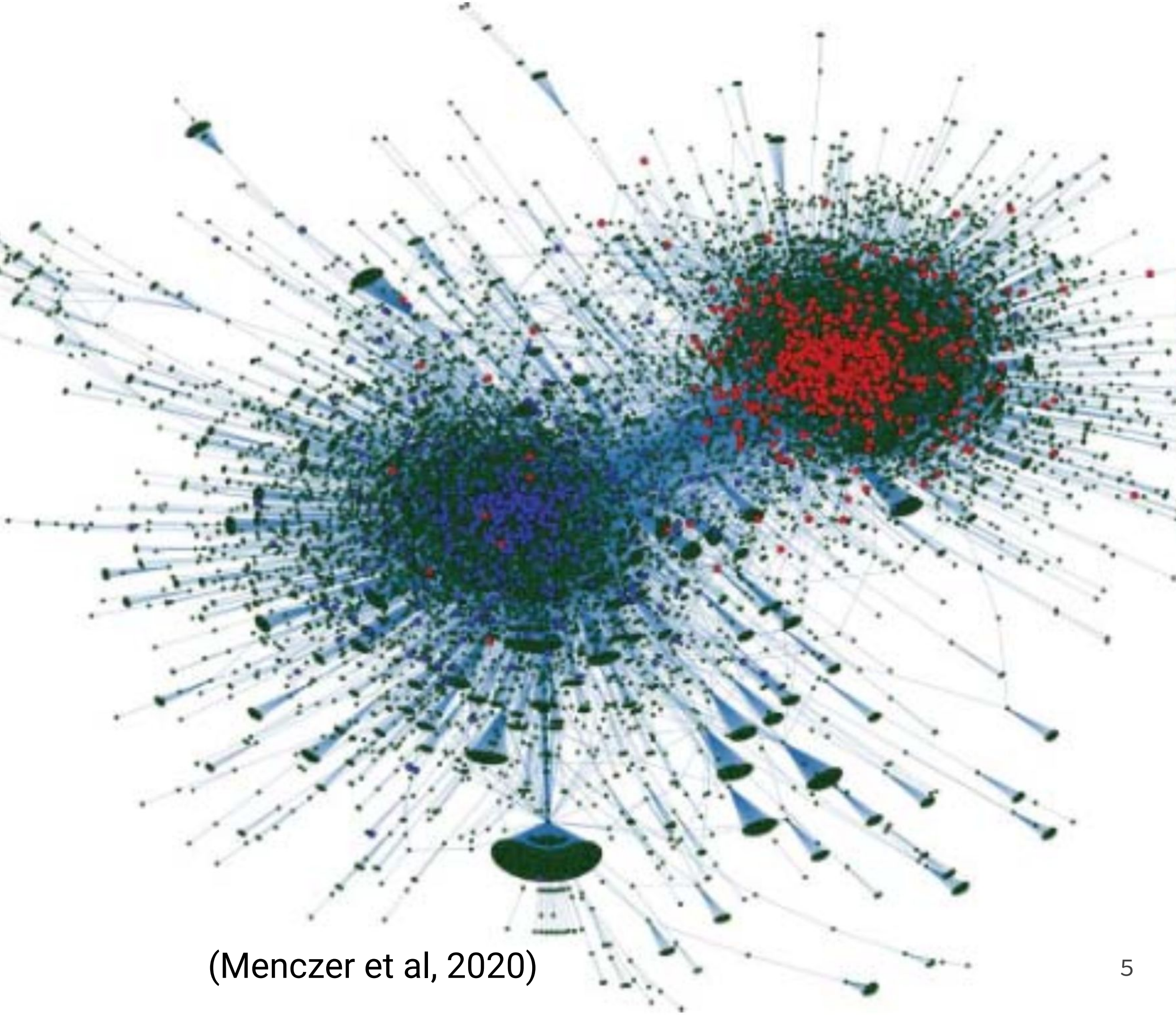
Further creative extensions:

- Building on the eigenvector-like centrality measure
- Insight: Fragility of neighbors increases my fragility
- Used Herfindahl index of concentration instead of node degree

Social networks are full of easy to spot “communities” (cohesive subgroups)

Twitter users

Retweet network of political hashtags on Twitter prior to the 2010 US election.



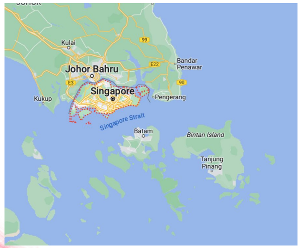
(Menczer et al, 2020)

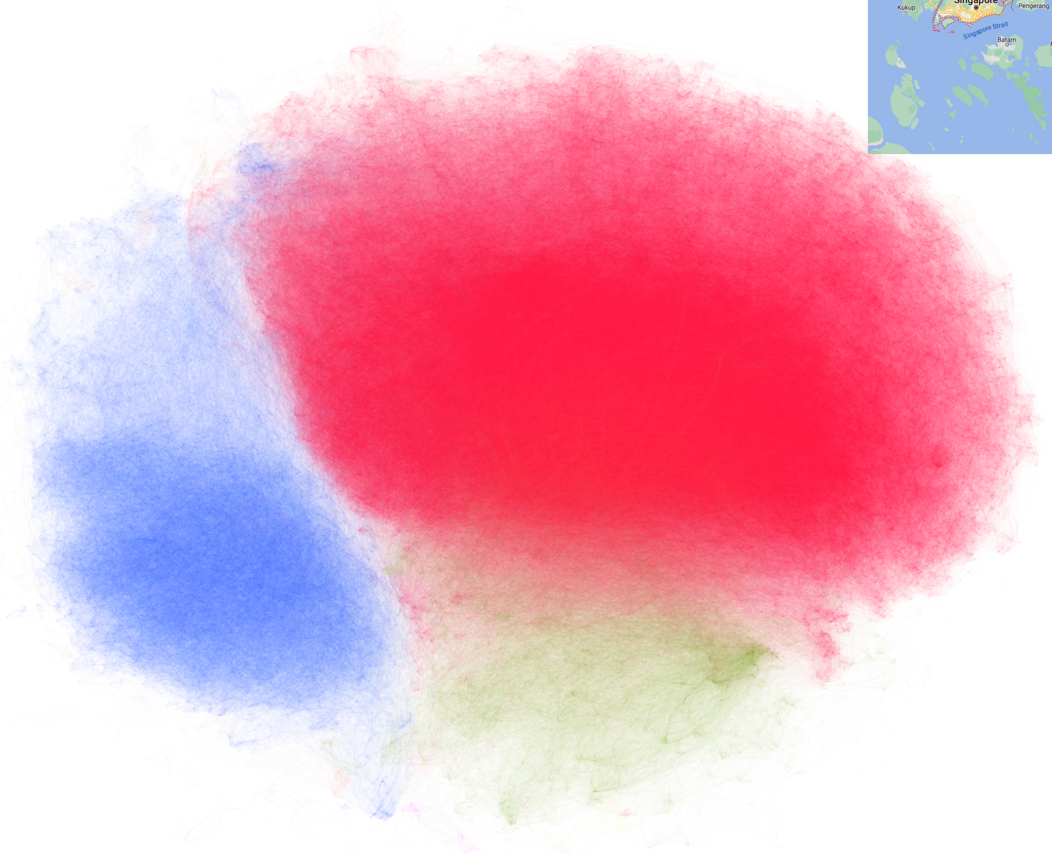
Twitter users

Bidirected @mention network among Singapore Twitter users

Colors based on community detection

Q: What attribute do you think the colors correspond to?





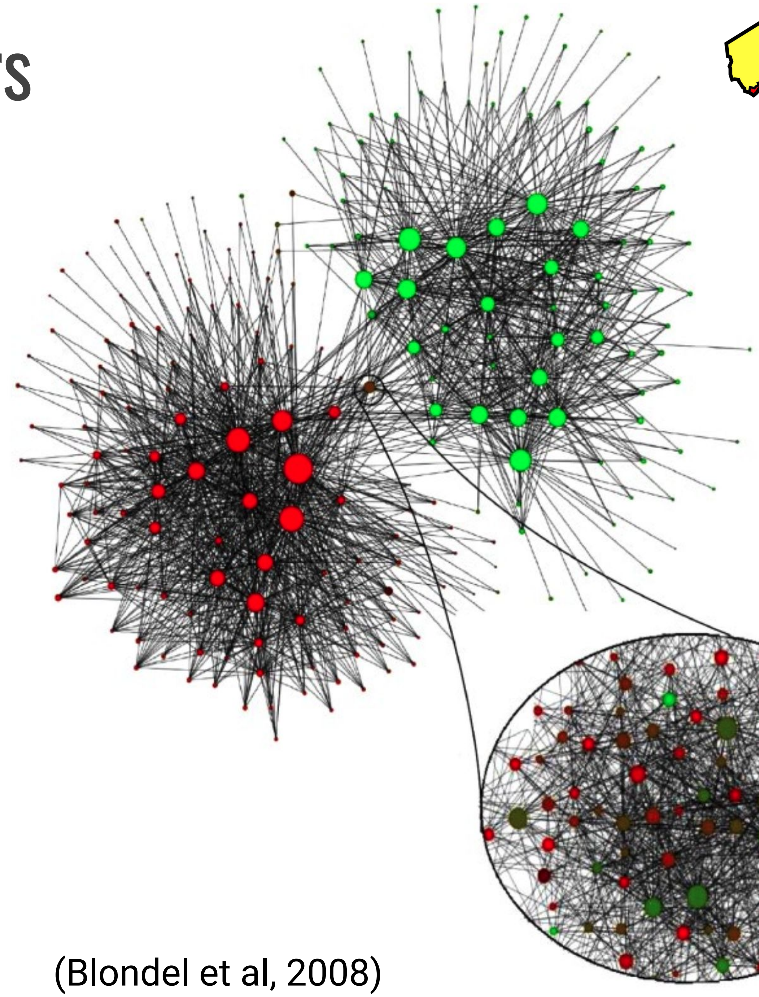
(Patrick Park, unpublished)

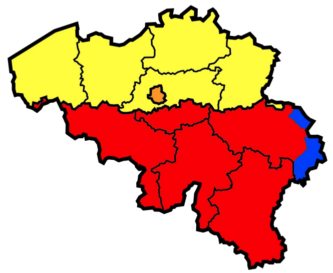
Belgian mobile phone users

The nodes correspond to communities.

The color represents the language spoken in the particular community: red for French and green for Dutch.

Bridge communities (Brussels) show less obvious language separation.





(Blondel et al, 2008)

Subgroups: easy to spot, but tricky to define

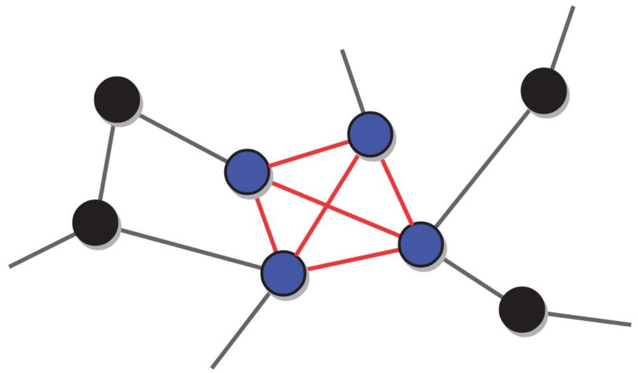
Social group is fundamental to humans

Yet a “group” lacks formal definition

- Too “obvious” to define
- But what is a group?

They come in all size, shapes, and forms

- Size (family, nation state)
- Intimacy (private vs. professional)
- Language
- Geography
- Means of production (capitalist vs. proletariat)
- …

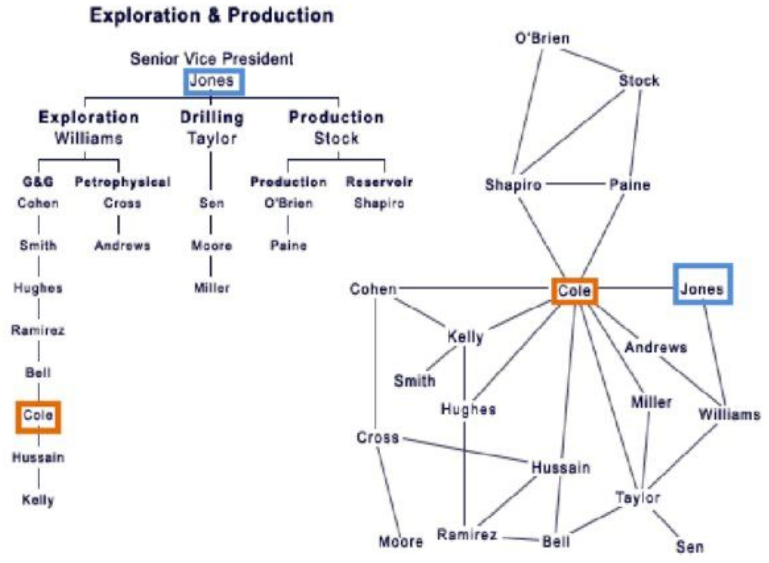


Subgroups: easy to spot, but tricky to define

The difficulty is apparent when you try to define groups top-down

- University C, Department X, Unit Z
- Member overlap: Department X and Y can share common members
- Informal groups: Some members in X have stronger ties to members in Y

Actual cohesion does not always form along formal groups

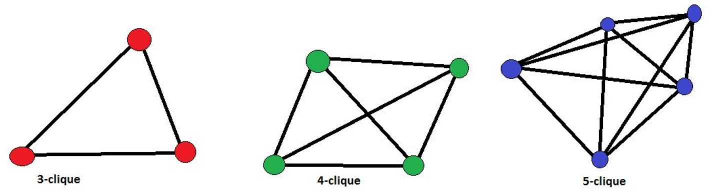


Source: Rob Cross, What is ONA? http://www.robcross.org/network_ona.htm

Subgroups: easy to spot, but tricky to define

The network approach develops Bottom-up approach to quantifying subgroups based on:

- Direct connections:
  - Clique: maximal subset of nodes with direct ties to one another

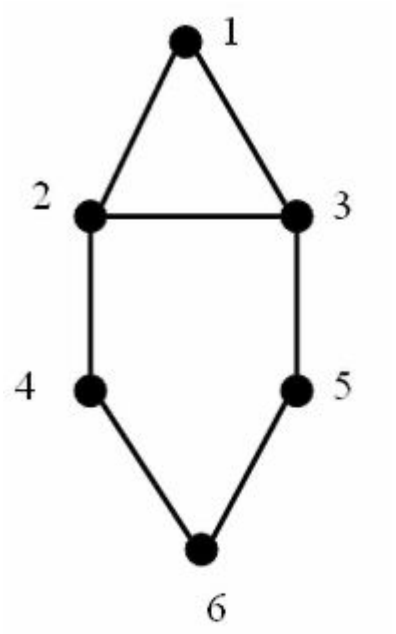


Subgroups: easy to spot, but tricky to define

The network approach develops Bottom-up approach to quantifying subgroups based on:

- Direct connections:
  - Clique: maximal subset of nodes with direct ties to one another
- Distance:
  - *n*-clan: Maximal subgraph of nodes that are within a path length *n* only through the nodes in that subset

2-clan: {2, 3, 4, 5, 6}



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- Distance:
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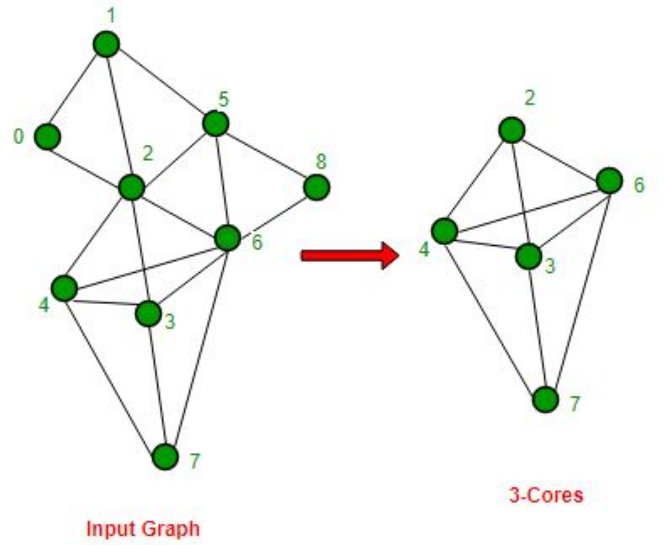
- Redundancy (many ways to reach others):
  - *k*-core: Maximal subgraph in which every node has edges to at least *k* other nodes in the subgraph

k-core



Subgroups: easy to spot, but tricky to define

The network approach develops Bottom-up approach to quantifying subgroups based on:

- Direct connections:
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- Distance:
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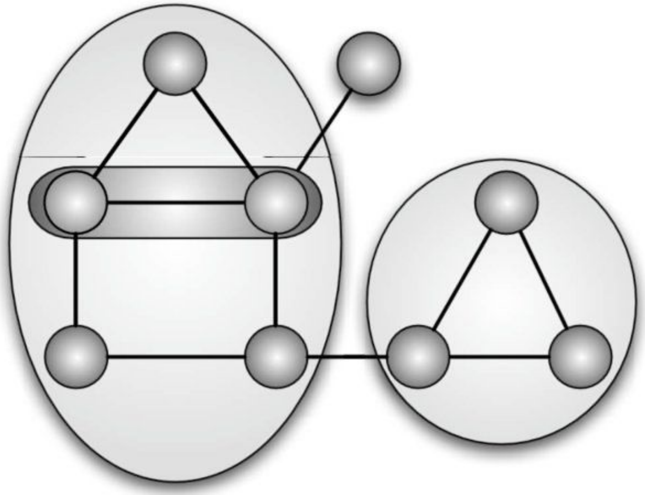
- Redundancy (many ways to reach others):
  - *k*-core: Maximal subgraph in which every node has edges to at least *k* other nodes in the subgraph
  - *k*-component: Every node has at least *k* non-overlapping paths to every node in the subgraph

k-components



Basic concepts

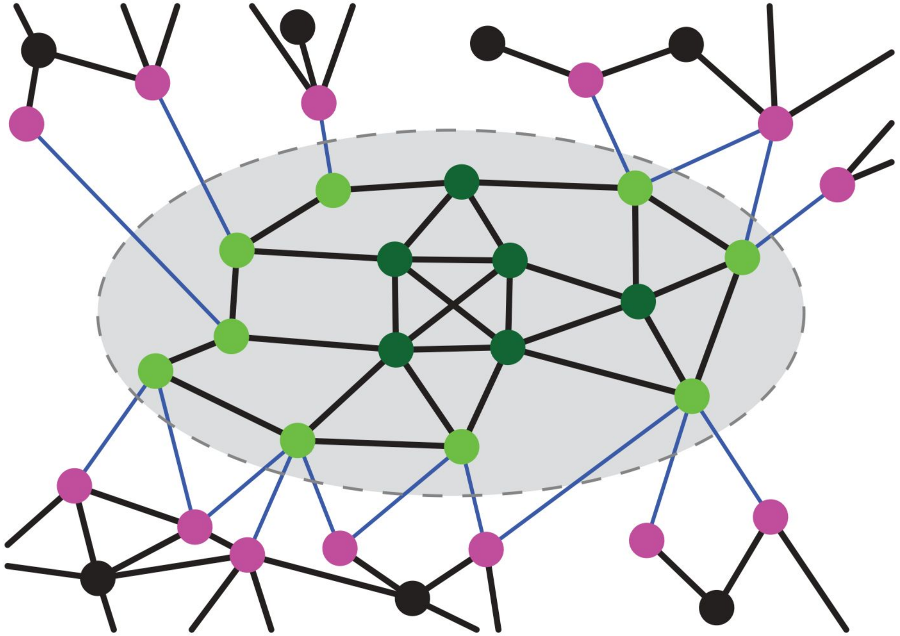
Internal links (black links)

Internal (black links) & external (blue links) degree of a node in the community (green nodes)

k_i^{int} k_i^{ext}

Community degree (sum of num neighbors of each internal node)

$$k_C = \sum_{i \in C} k_i.$$



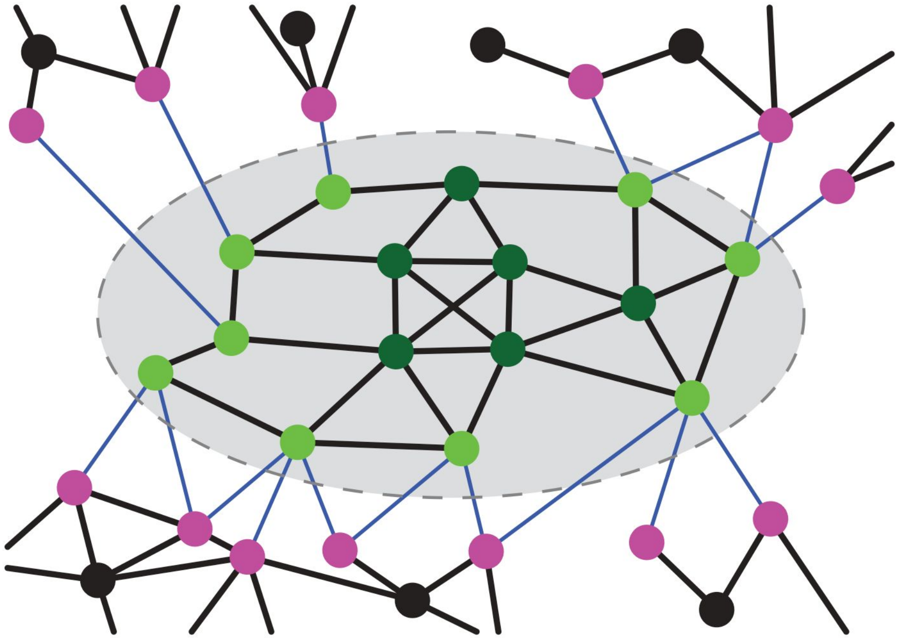
Recall

The maximum number of links in an undirected network with N nodes:

?

The density of a network with N nodes and L links:

?



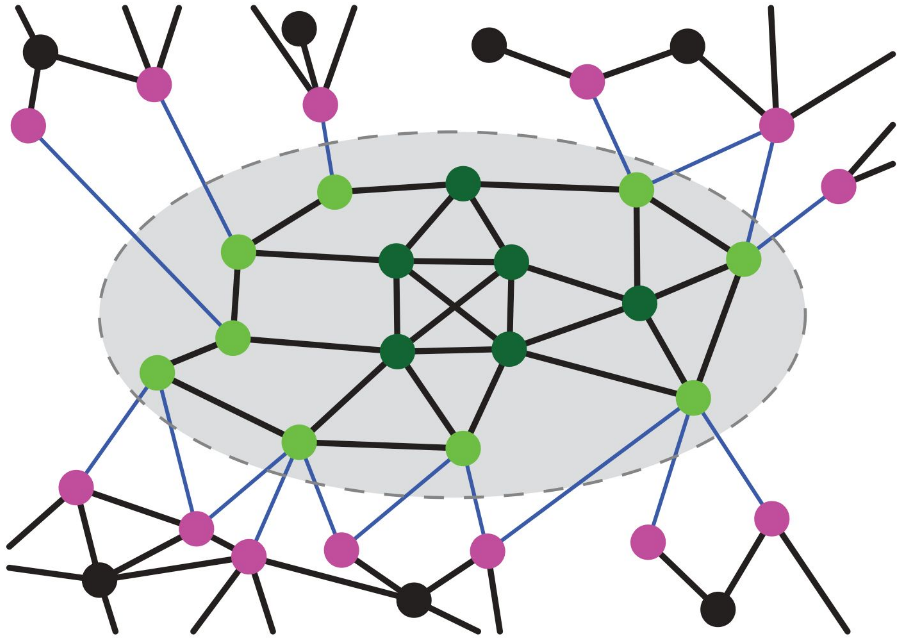
Recall

The maximum number of links in an undirected network with N nodes:

$$L_{max} = \binom{N}{2} = N(N-1)/2.$$

The density of a network with N nodes and L links:

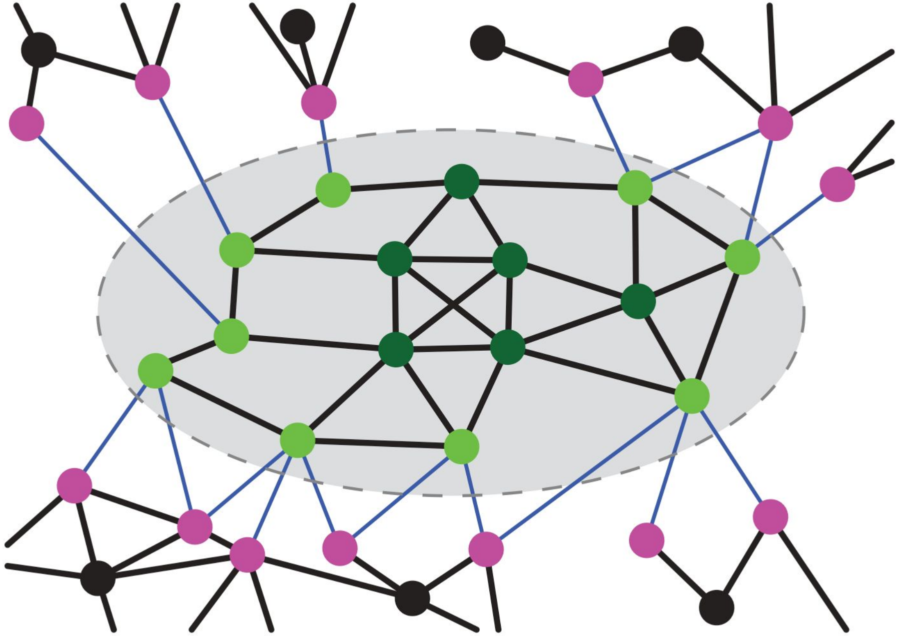
$$d = L/L_{max} = \frac{2L}{N(N-1)}$$



Basic concepts

Internal link density:

$$\delta_C^{int} = \frac{L_C}{\binom{N_C}{2}} = \frac{2L_C}{N_C(N_C - 1)}.$$



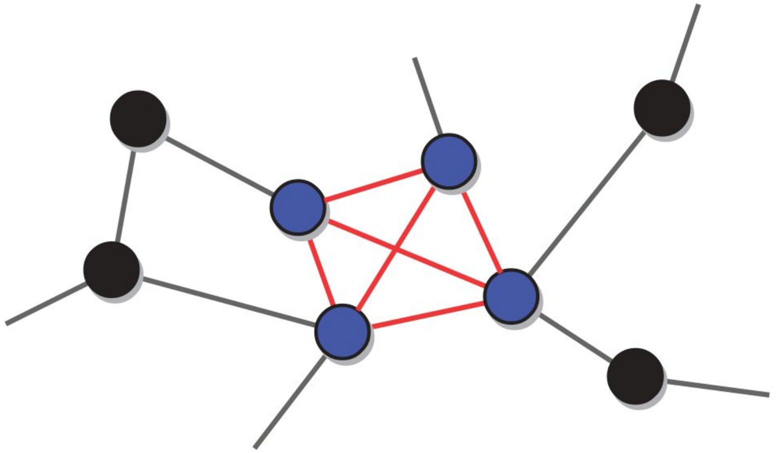
Intuition: Nodes within a “community” have higher likelihood of connecting to each other than to nodes from other “communities.”
$\rightarrow$ high “cohesion,” high “separation”

Aside: Cliques have high cohesion, but aren't realistic communities

Real communities aren't as dense as cliques.

In real communities some nodes are more important than others.

Better: The number of internal links should be larger than the number of external links.



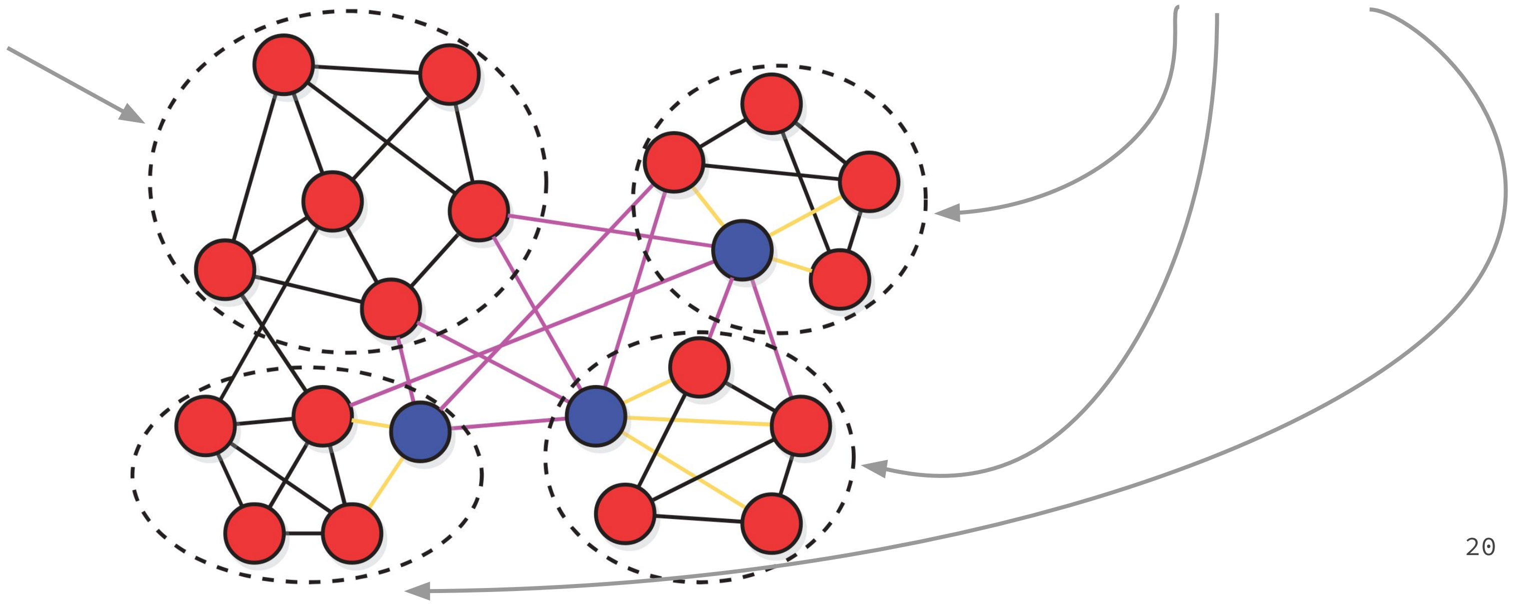
“Strong” vs “weak” communities

Strong

The internal degree of each node exceeds its external degree towards other communities.

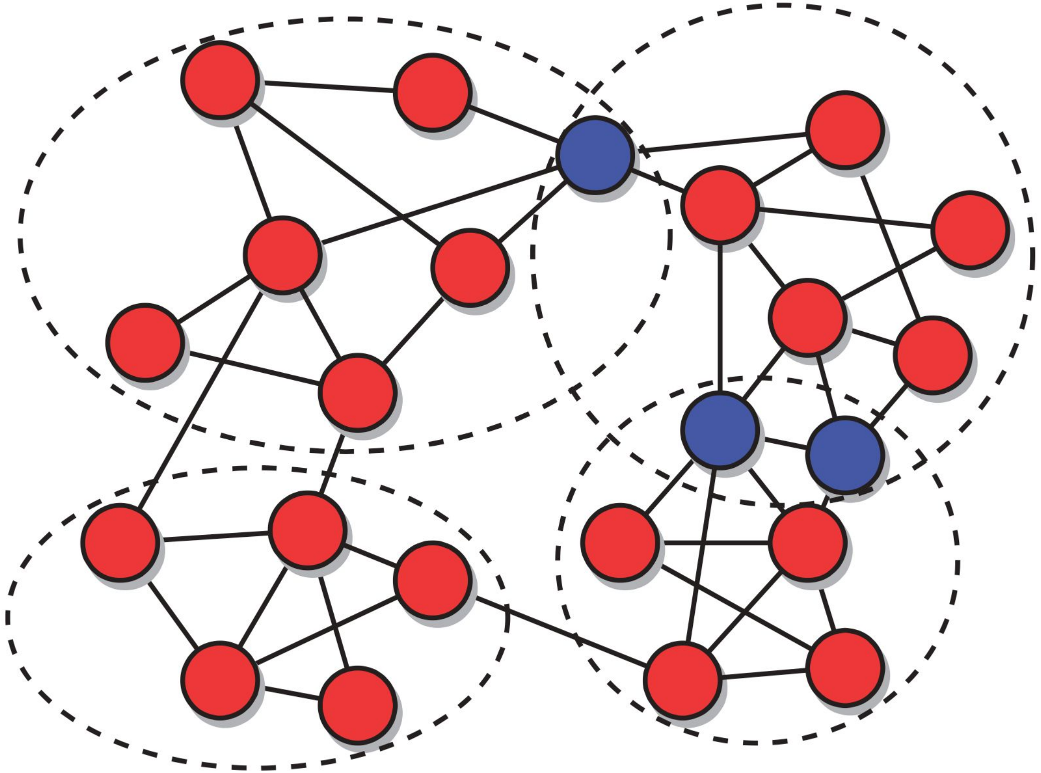
Weak

The sum of internal degrees of all nodes exceeds the sum of their external degrees in other communities.

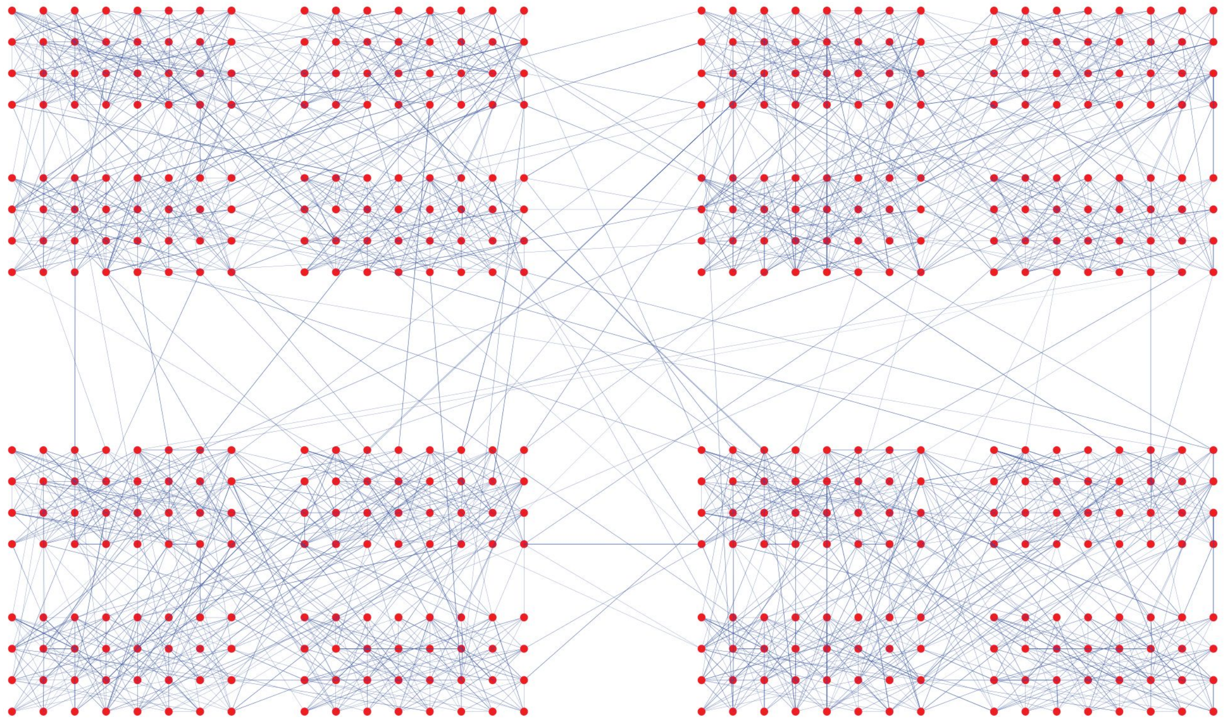


strong → weak

The communities in many real-world networks overlap



Partitions can be hierarchical when the network has multiple levels of organization

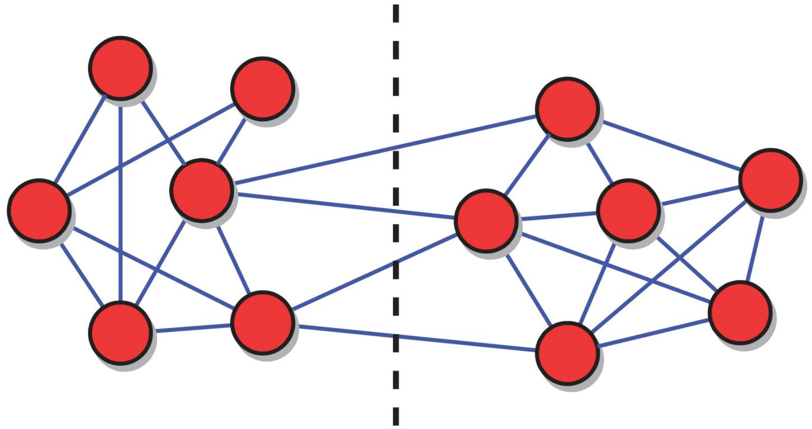


So, how to find the communities?

1. Graph partitioning – old problem

Min-cut graph bisection doesn’t quite work

Trivial solution to minimizing cut size: single cluster containing the entire network gives cut size of zero.



→ Need to specify the number of clusters beforehand.

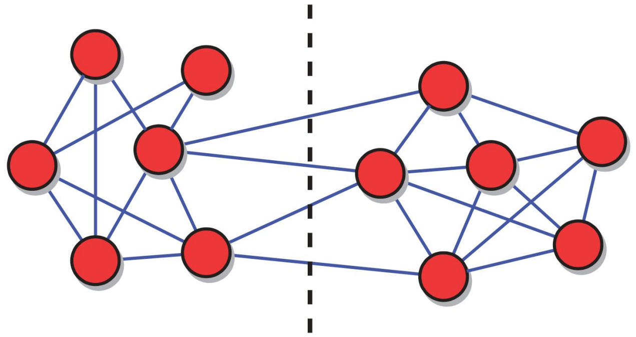
(Also need to specify size of each cluster beforehand. Example: one leaf vs all other nodes)

Kernighan-Lin graph bisection algorithm

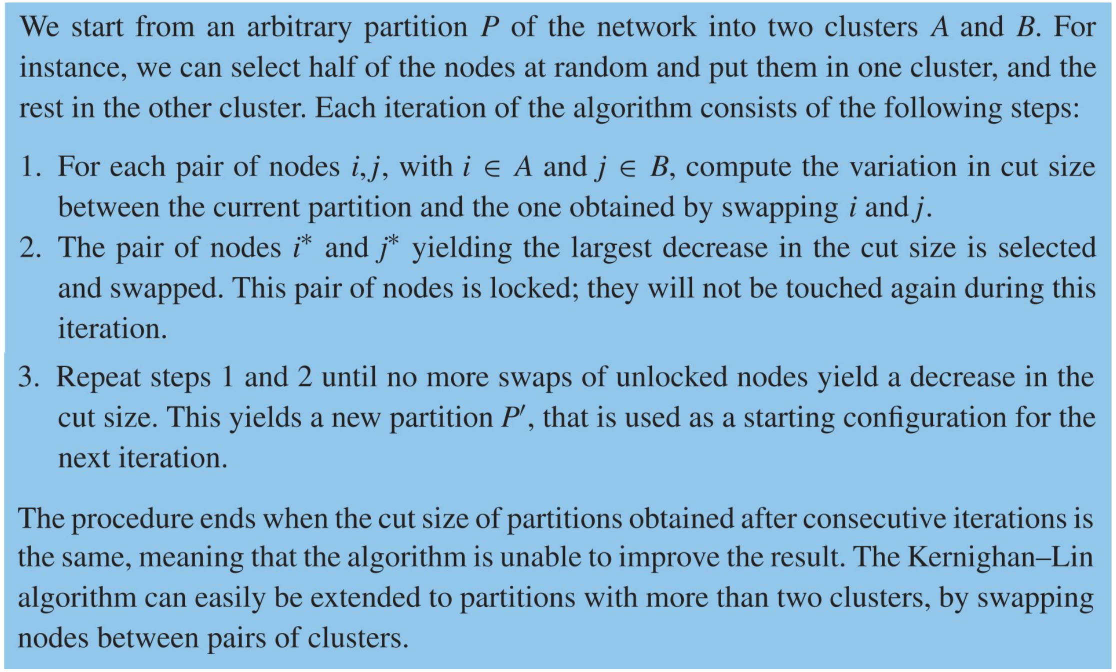
We start from an arbitrary partition P of the network into two clusters A and B. For instance, we can select half of the nodes at random and put them in one cluster, and the rest in the other cluster. Each iteration of the algorithm consists of the following steps:

1. For each pair of nodes i,j, with $i \in A$ and $j \in B$, compute the variation in cut size between the current partition and the one obtained by swapping i and j.
2. The pair of nodes i^* and j^* yielding the largest decrease in the cut size is selected and swapped. This pair of nodes is locked; they will not be touched again during this iteration.
3. Repeat steps 1 and 2 until no more swaps of unlocked nodes yield a decrease in the cut size. This yields a new partition P', that is used as a starting configuration for the next iteration.

The procedure ends when the cut size of partitions obtained after consecutive iterations is the same, meaning that the algorithm is unable to improve the result. The Kernighan–Lin algorithm can easily be extended to partitions with more than two clusters, by swapping nodes between pairs of clusters.



Kernighan-Lin graph bisection algorithm



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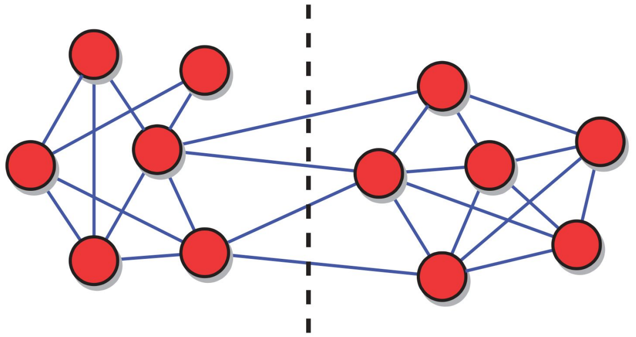


Greedy, risks getting stuck in local optima.

Not bad, but we can do better.
Clusters identified via network partitioning are well-separated but not necessarily cohesive.

2. Clustering – also old problem

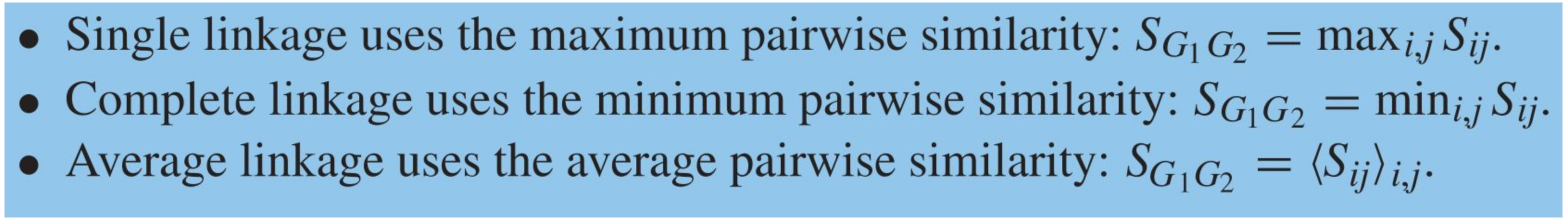
The main ingredient is a similarity measure between nodes

A classic example is structural equivalence, which expresses the similarity between the neighborhoods of a pair of nodes.

$$S_{ij}^{SE} = \frac{\text{number of neighbors shared by } i \text{ and } j}{\text{total number of nodes neighboring only } i\text{, only } j\text{, or both}} .$$

We also need to define similarity for groups of nodes

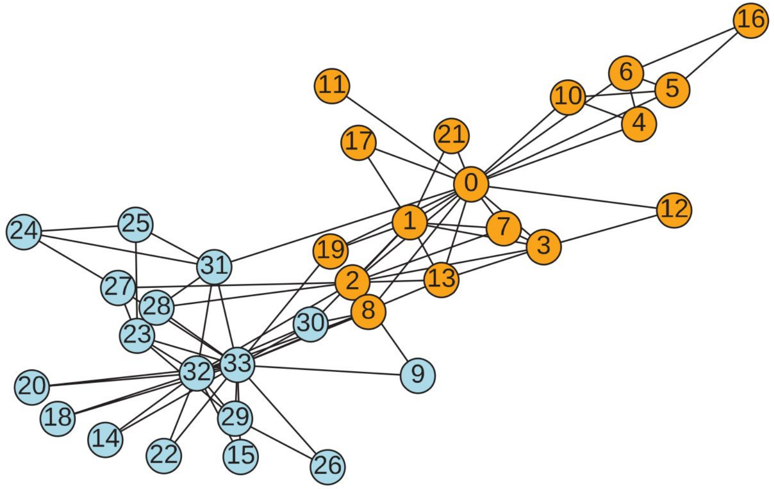
Given a node similarity measure S and two groups of nodes G1 and G2:



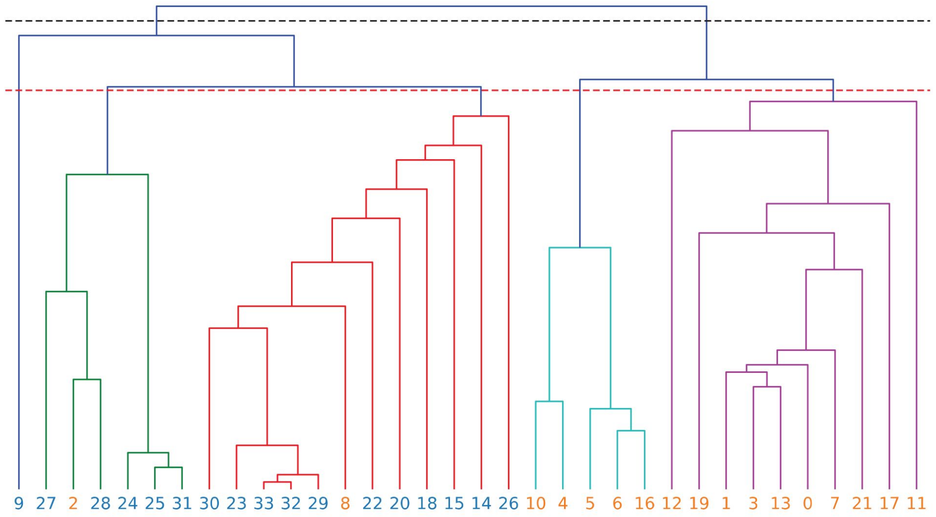
- Single linkage uses the maximum pairwise similarity: $S_{G_1 G_2} = \max_{i,j} S_{ij}$.
- Complete linkage uses the minimum pairwise similarity: $S_{G_1 G_2} = \min_{i,j} S_{ij}$.
- Average linkage uses the average pairwise similarity: $S_{G_1 G_2} = \langle S_{ij} \rangle_{i,j}$.

Next we can apply agglomerative hierarchical clustering

Start from the trivial partition into N groups. At each step, merge the pair of groups with the largest similarity. Repeat until all nodes are in the same group.



Zachary's karate club network.
Node 0: instructor. Node 33: club president



Next we can apply agglomerative hierarchical clustering

Start from the trivial partition into N groups. At each step, merge the pair of groups with the largest similarity. Repeat until all nodes are in the same group.

Complexity:

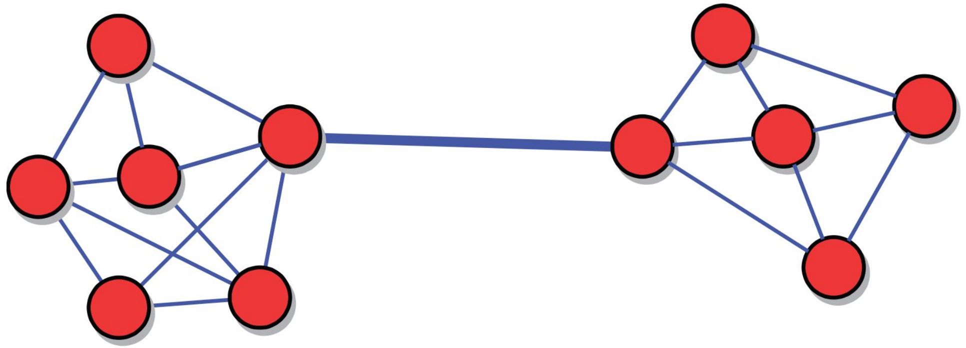
- We need to compare N^2 node pairs to compute pairwise similarity.
- Group similarity requires us to determine in each step the distance of the new cluster to all other clusters. Doing this N times requires $0(N^2)$ calculations.
- The construction of the dendrogram can be performed in 0(NlogN) steps.
- Total $0(N^2)$.

As many partitions as there are nodes → Unclear which partition is meaningful for the given network. Plus, rather slow.

3. Community detection

3.1. Bridge removal

Key idea: Find links with high betweenness and remove them.



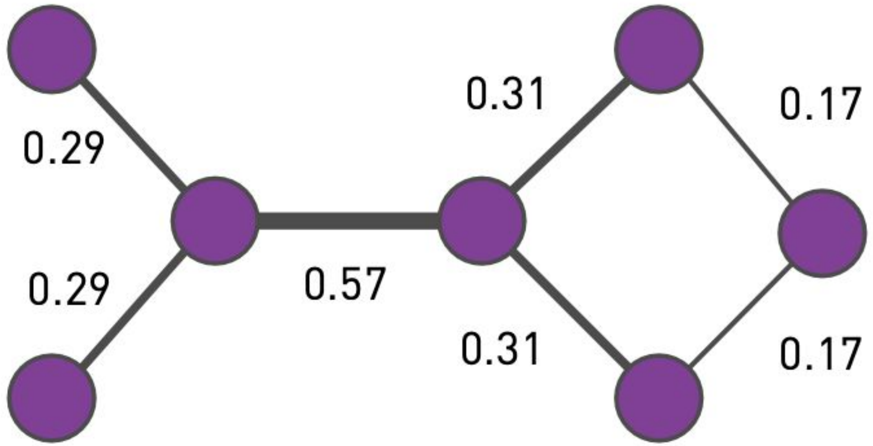
Link betweenness defined similarly to node betweenness centrality in previous lecture – fraction of shortest paths that run through that link.

Link betweenness should be higher for bridges than for links inside a cluster.

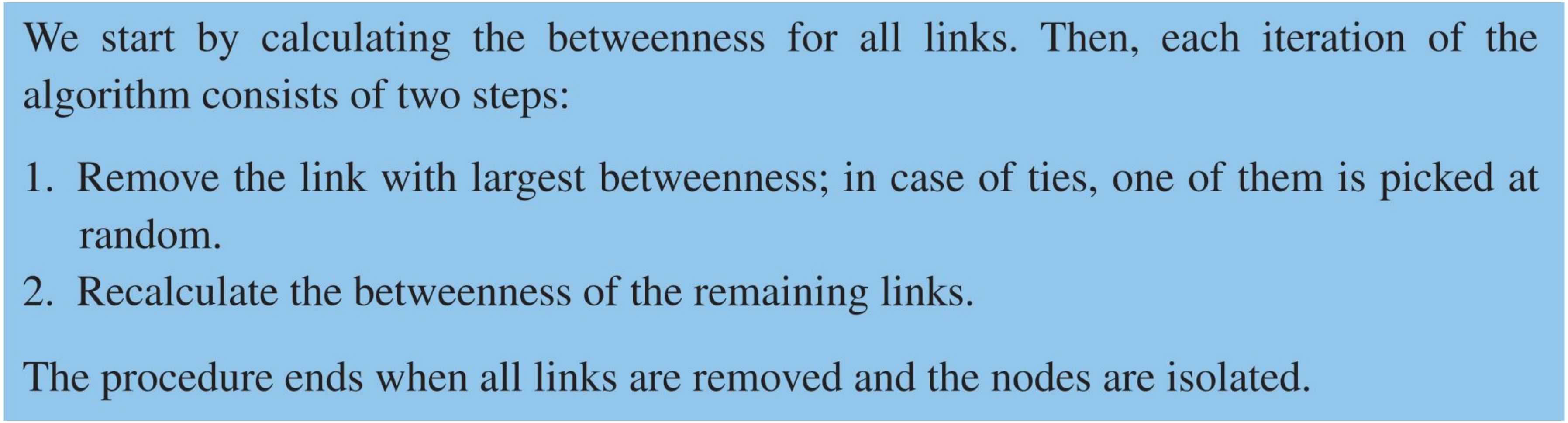
Example calculating link betweenness

Inter-community links, like the central link in the figure with x_{ij}=0.57, have large betweenness.

The calculation of link betweenness scales as 0(LN), or 0(N^2) for a sparse network.



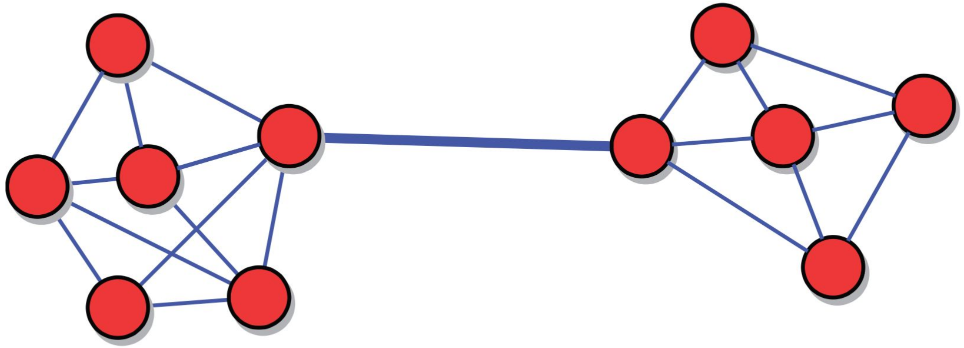
Girvan-Newman algorithm (similar to hierarchical clustering)



We start by calculating the betweenness for all links. Then, each iteration of the algorithm consists of two steps:

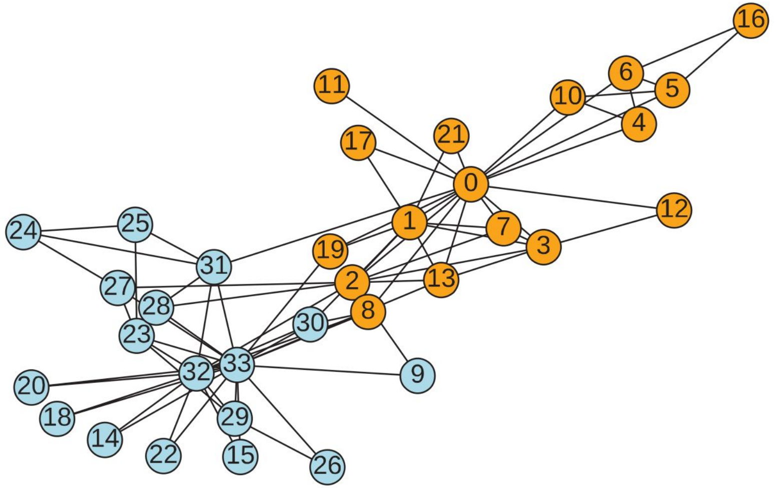
1. Remove the link with largest betweenness; in case of ties, one of them is picked at random.
2. Recalculate the betweenness of the remaining links.

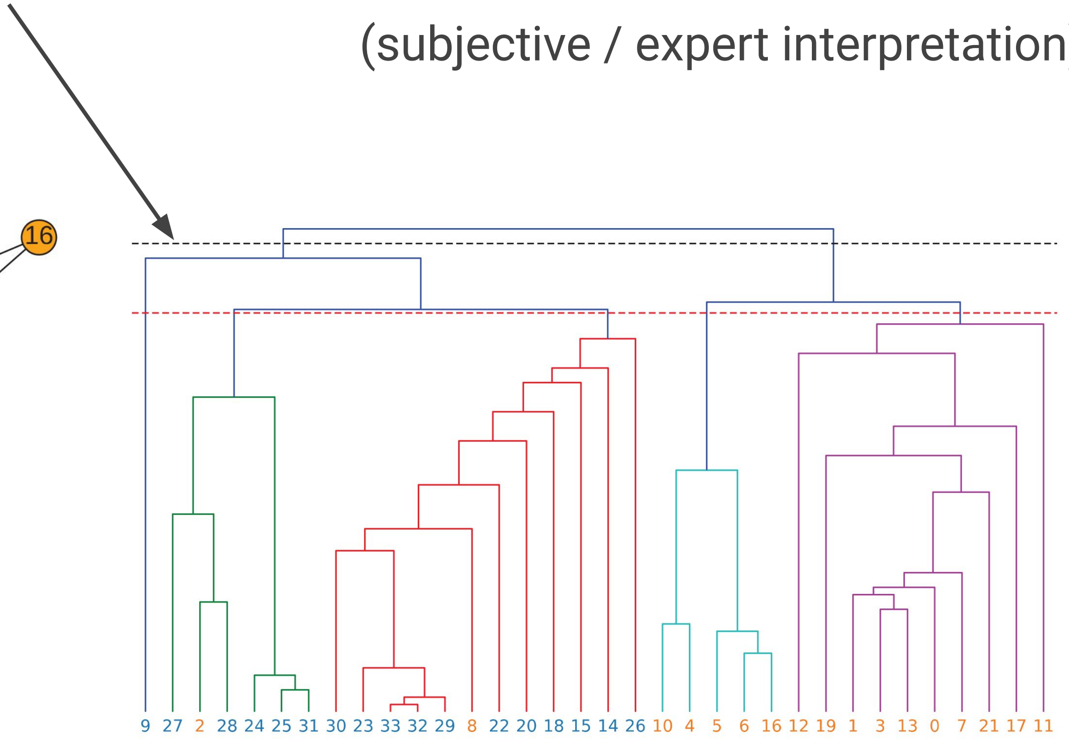
The procedure ends when all links are removed and the nodes are isolated.



Girvan-Newman algorithm on Zachary's karate club: 2 clusters

(subjective / expert interpretation)





Girvan-Newman algorithm - reflections

Slow – must recompute the betweenness of all links each iteration.

- Step 2 introduces an additional factor L in the running time, hence the algorithm scales as $0(L^2N)$, or $0(N^3)$ for a sparse network.

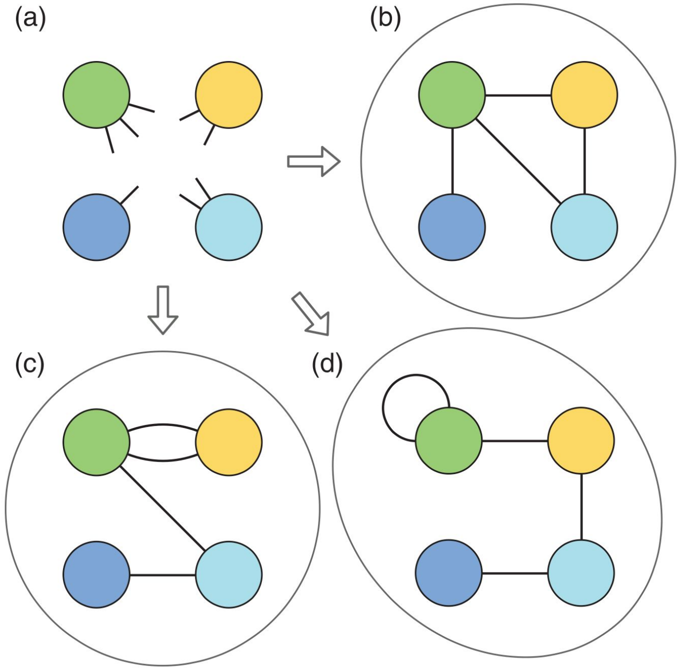
Improvement: recompute betweenness only within the connected component including the last removed link.

We still need a measure of the quality of a partition.

Modularity

The difference between the number of links internal to all clusters and the expected equivalent number in a randomized network.

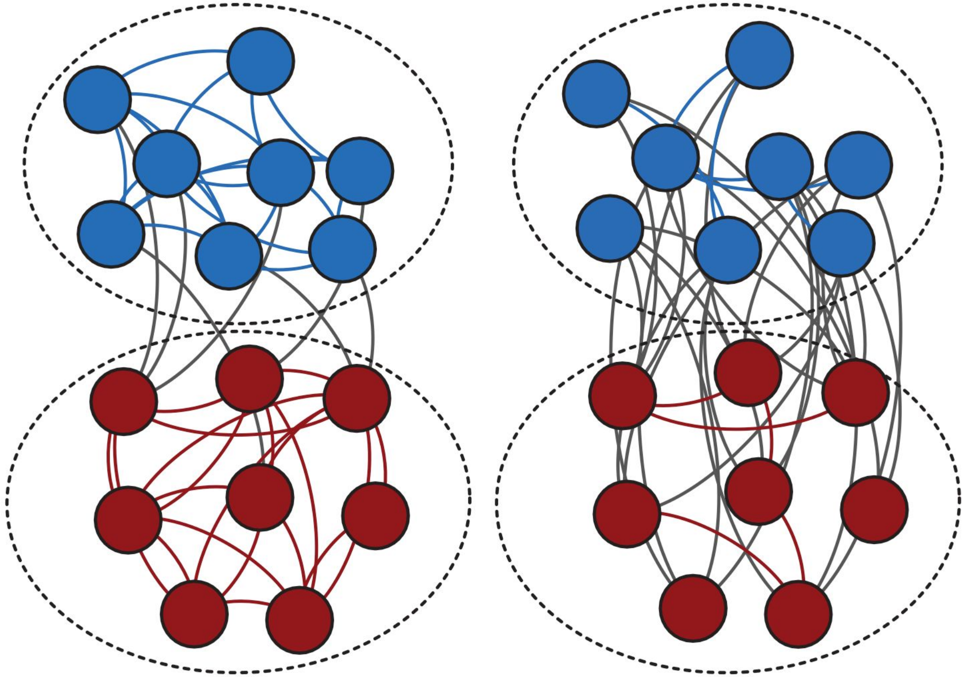
Randomization strategy: maintain number of nodes and degree sequence, shuffle links.



Modularity

Left network: visible community structure (high modularity).

Right network: degree-preserving randomization – fewer internal links and more links between the subnets.



Modularity

The modularity of a partition in an undirected, unweighted network:

$$Q = \frac{1}{L}\sum_{C}\left(L_C - \frac{k_C^2}{4L}\right)$$

Lc is the number of internal links in cluster C, kc is the total degree of nodes in C.

kc (total num stubs attached to nodes in C) stays constant in each randomization, by construction.

The probability of selecting one of these stubs at random is: $k_C/2L$

The probability of picking a pair of stubs from C at random is: $\frac{k_C}{2L} \cdot \frac{k_C}{2L} = \frac{k_C^2}{4L^2}$

Modularity

The modularity of a partition in an undirected, unweighted network:

$$Q = \frac{1}{L} \sum_{C} \left(L_C - \frac{k_C^2}{4L} \right).$$

What happens when there is a single cluster?

Can Q ≥ 1?

Can Q < 0?

Modularity

The modularity of a partition in an undirected, unweighted network:

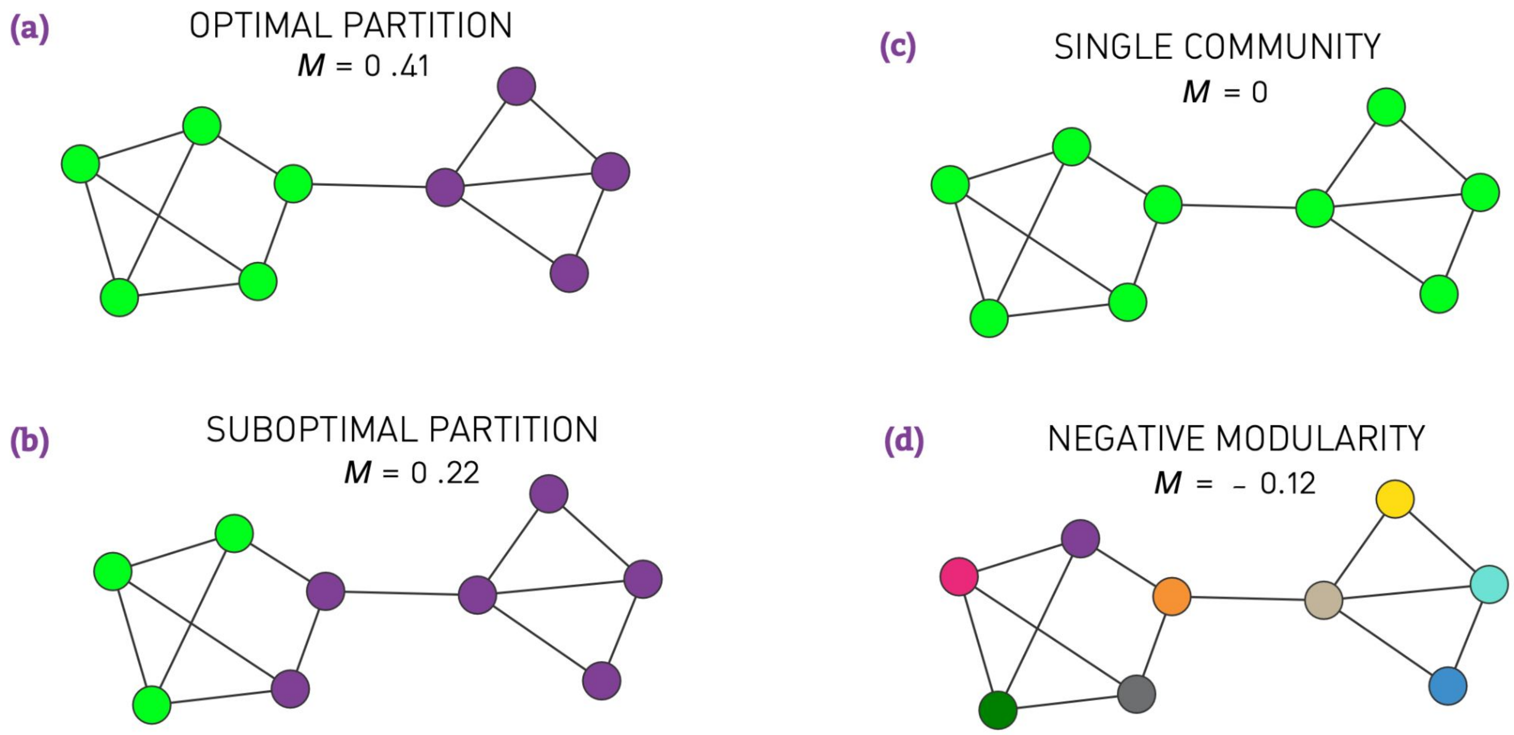
$$Q = \frac{1}{L} \sum_{C} \left(L_C - \frac{k_C^2}{4L} \right)$$

What happens when there is a single cluster? → Q = 0 ($L_C = L$, $k_C = 2L$)

Can Q ≥ 1? → No ($Q_{max} = (\sum_C L_C)/L$)

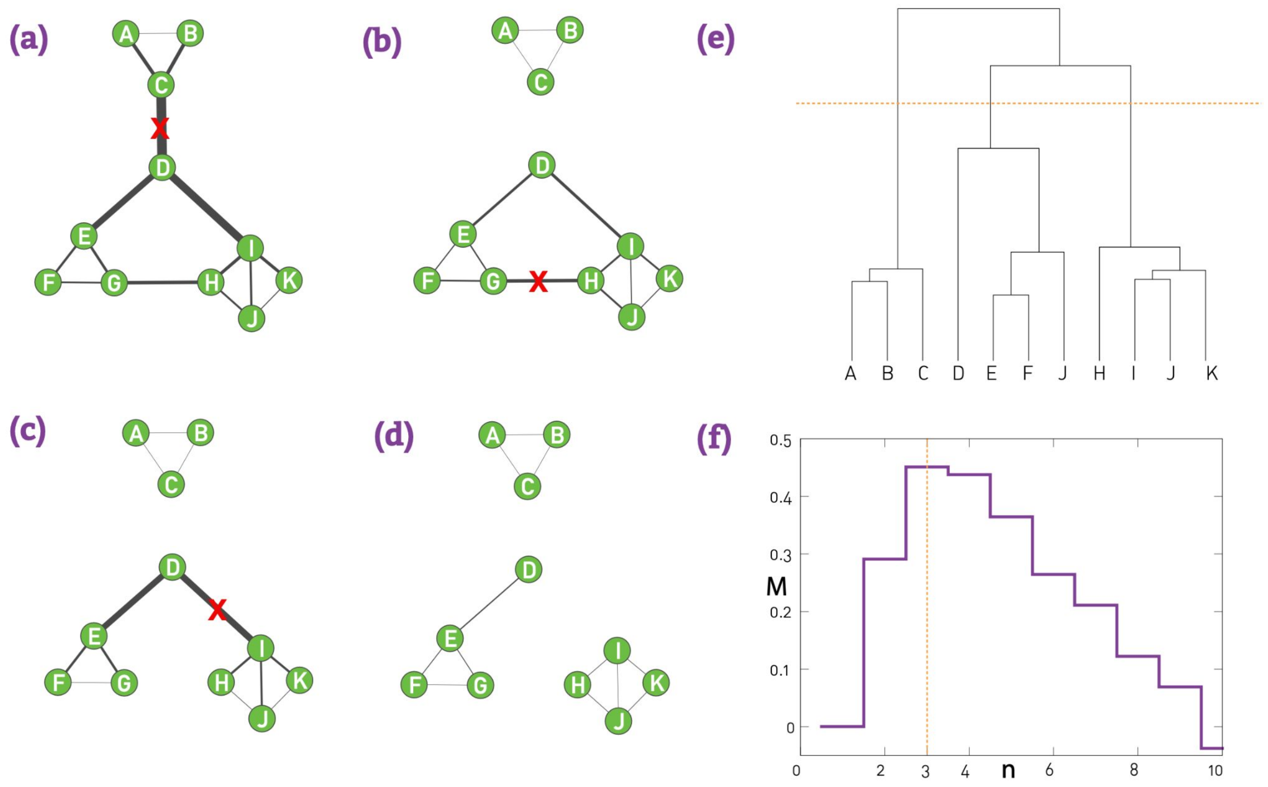
Can Q < 0? → Yes (partition into N singletons: $L_C = 0$)

The higher the modularity for a partition, the better the corresponding community structure

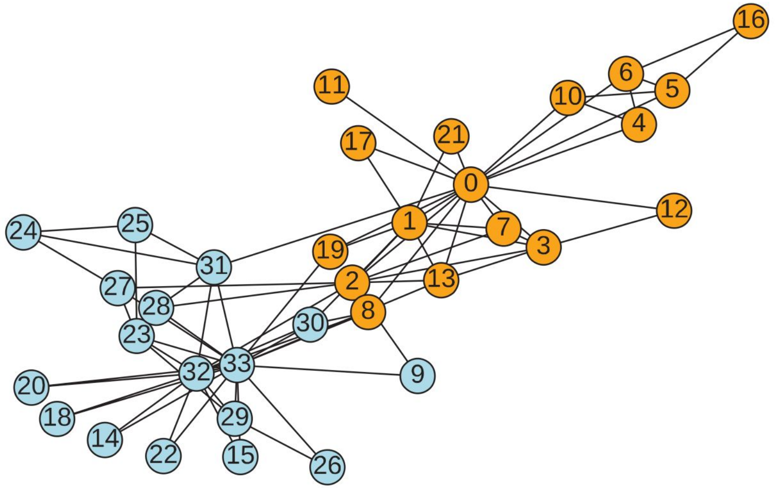


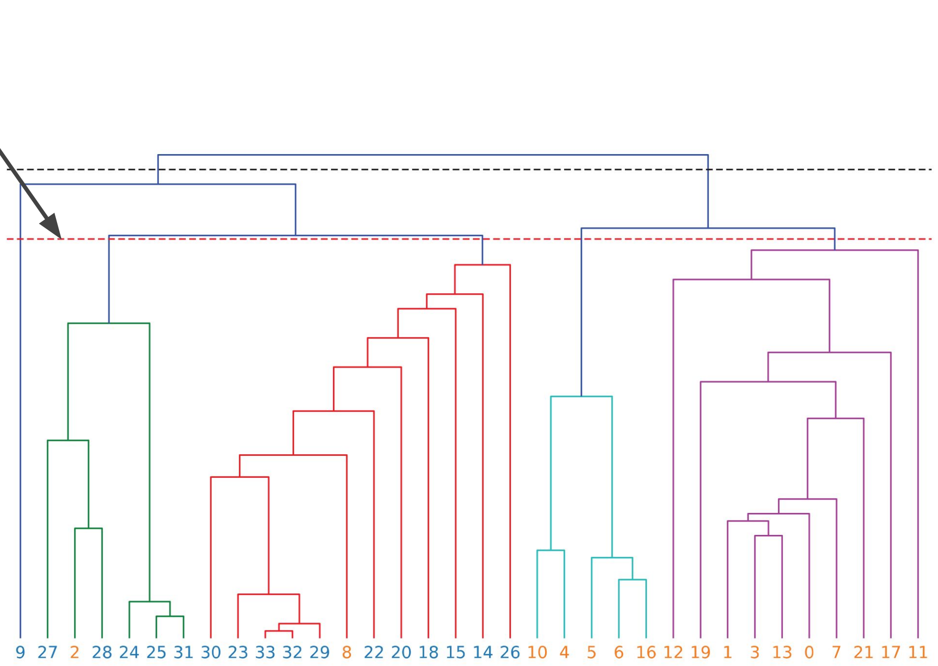
Use modularity to decide which partition predicted by a hierarchical method offers the best community structure

Select the one for which M is maximal!



Largest modularity value in Zachary's Karate Club: five clusters





3.2. Modularity optimization - Greedy algorithm

1. Assign each node to a community of its own, starting with N communities of single nodes.

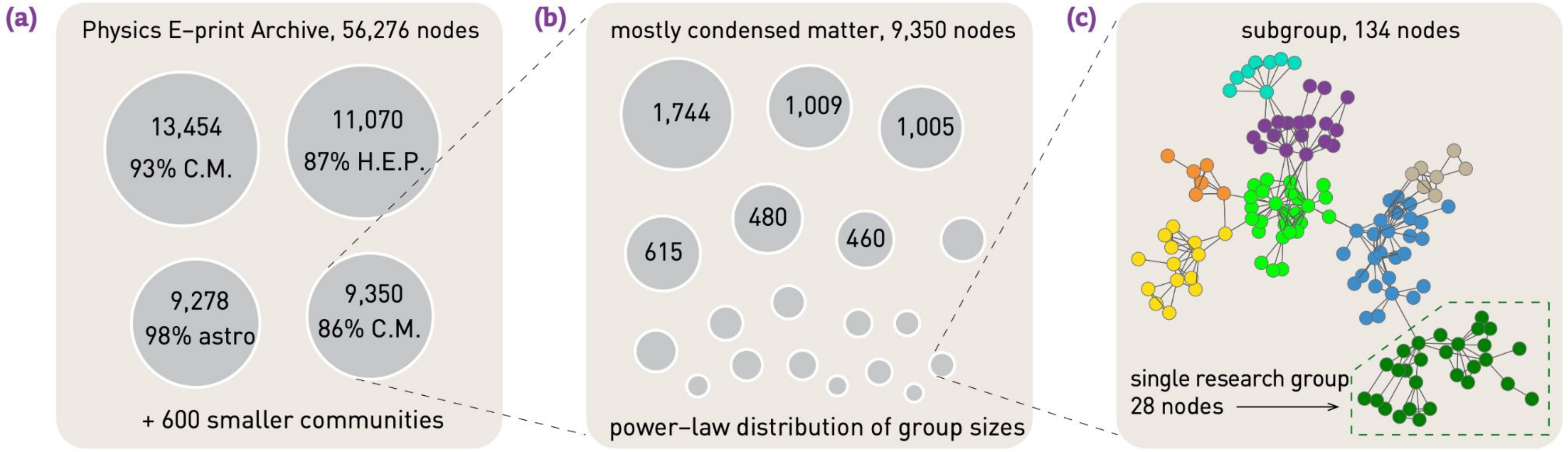
2. Inspect each community pair connected by at least one link and compute the modularity difference ΔM obtained if we merge them. Identify the community pair for which ΔM is the largest and merge them. Note that modularity is always calculated for the full network.

3. Repeat Step 2 until all nodes merge into a single community, recording M for each step.

4. Select the partition for which M is maximal.

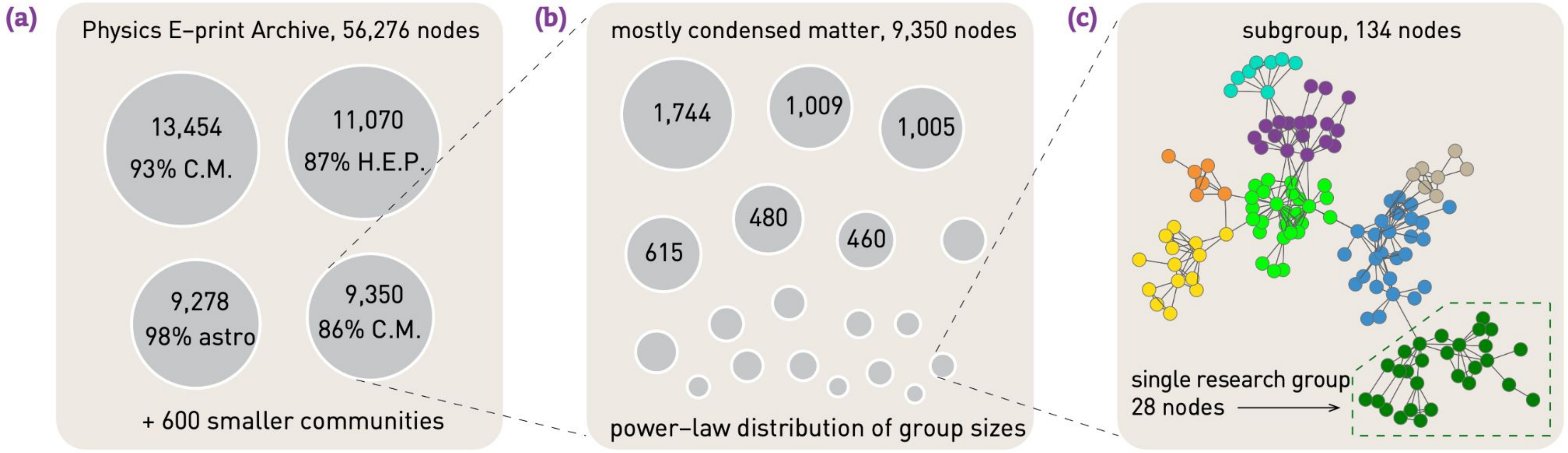
Clustering physicists using the greedy algorithm

(a) The greedy algorithm predicts four large communities, each composed primarily of physicists of similar interest.



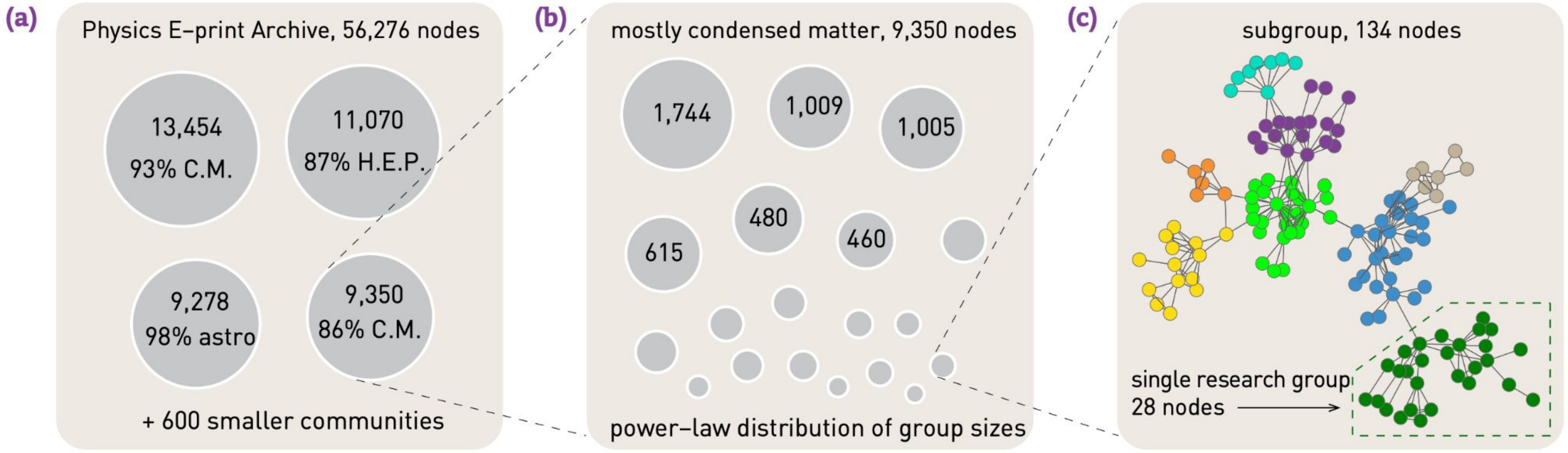
Clustering physicists using the greedy algorithm

(b) We can identify subcommunities by applying the greedy algorithm to each community, treating them as separate networks.



Clustering physicists using the greedy algorithm

(c) One of these smaller communities is further partitioned, revealing individual researchers and the research groups they belong to.

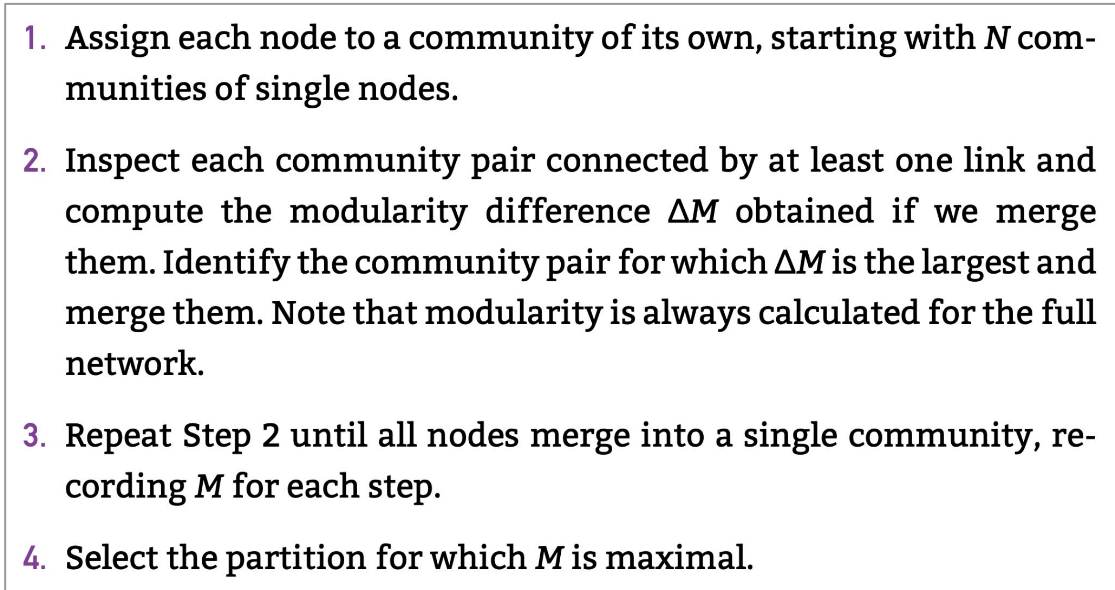


Complexity analysis of the greedy algorithm

The calculation of each ΔM can be done in constant time → Step 2 takes $O(L)$ computations.

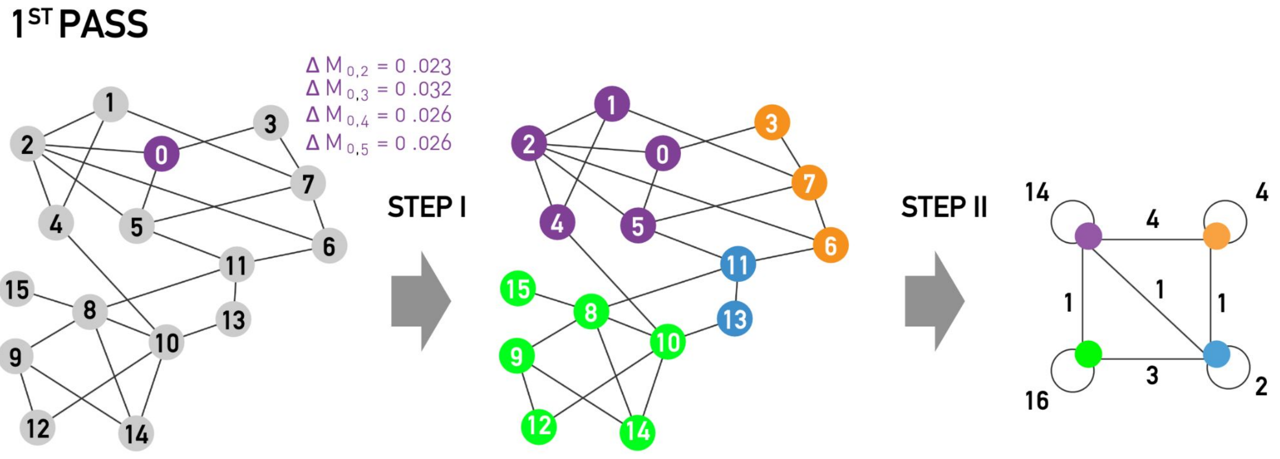
After deciding which communities to merge, the update of the matrix can be done in a worst-case time $O(N)$.

Since the algorithm requires $N-1$ community mergers, its complexity is $O[(L + N)N]$, or $O(N^2)$ on a sparse graph.



1. Assign each node to a community of its own, starting with N communities of single nodes.
2. Inspect each community pair connected by at least one link and compute the modularity difference ΔM obtained if we merge them. Identify the community pair for which ΔM is the largest and merge them. Note that modularity is always calculated for the full network.
3. Repeat Step 2 until all nodes merge into a single community, recording M for each step.
4. Select the partition for which M is maximal.

The Louvain algorithm



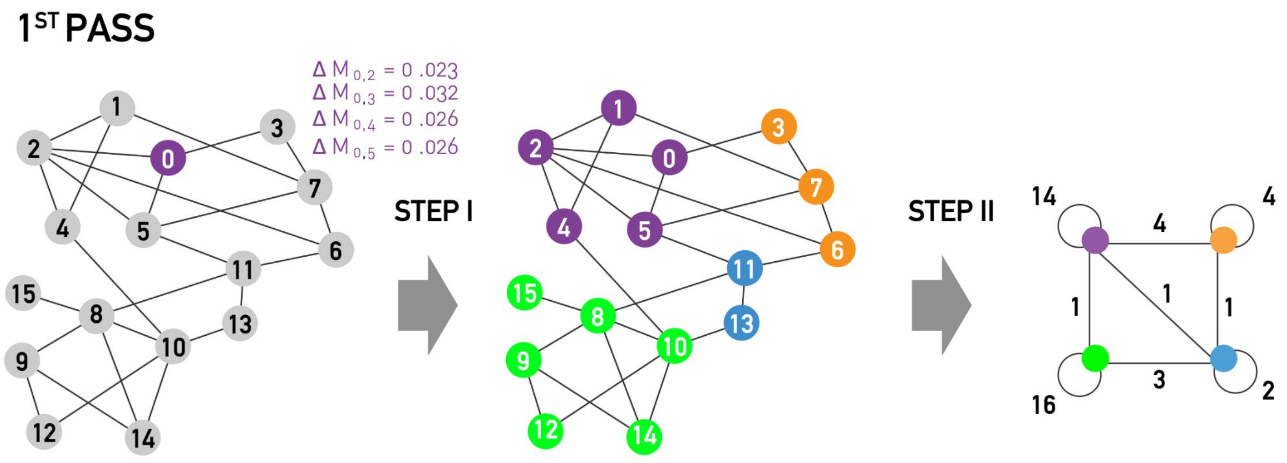
Step I: Modularity is optimized by local changes.

Choose a node (e.g., 0) and calculate the change in modularity if the node joins the community of its immediate neighbors.

→ Node 0 will join node 3.

Repeat for each node.

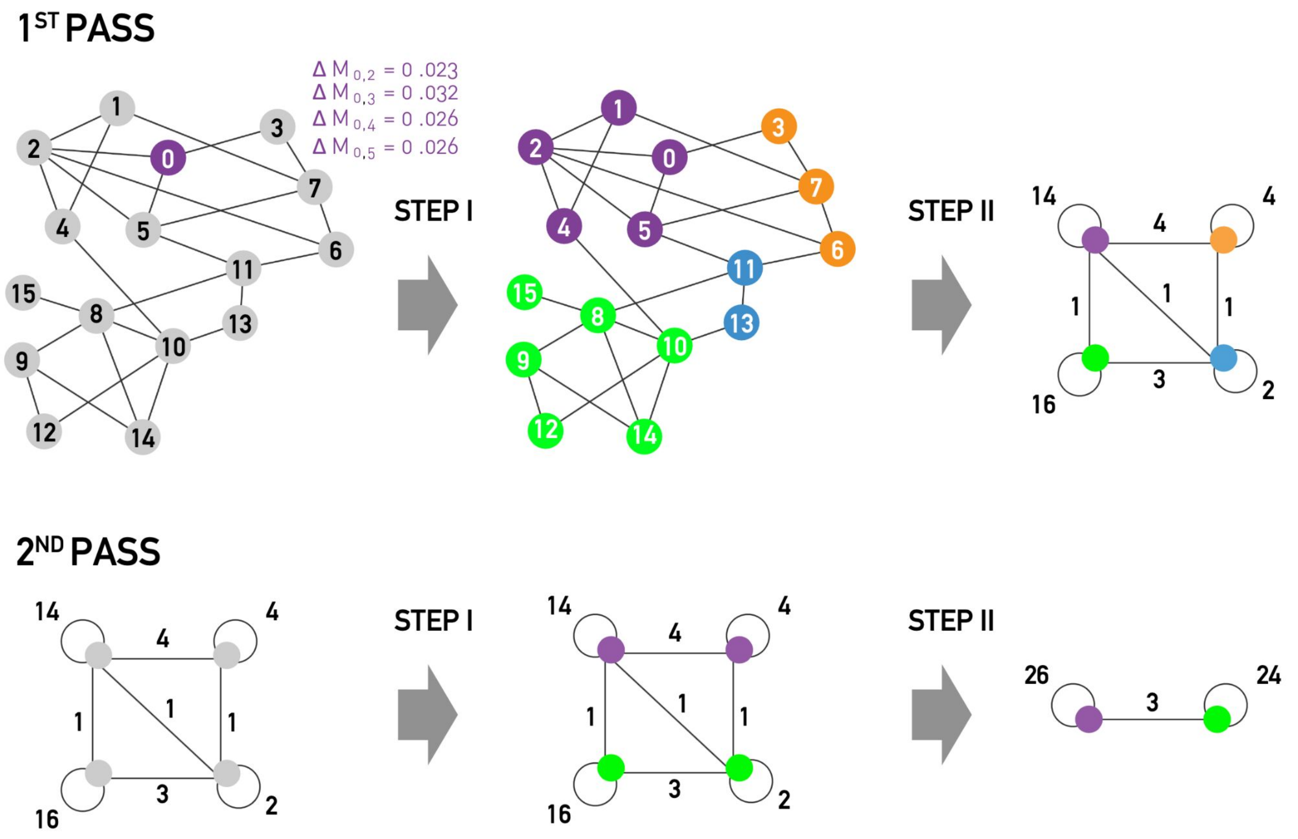
The Louvain algorithm



Step II: Aggregate the communities in Step I by merging nodes belonging to the same community into a single supernode.

This process will generate self-loops, corresponding to links between nodes in the same community that are now merged into a single node.

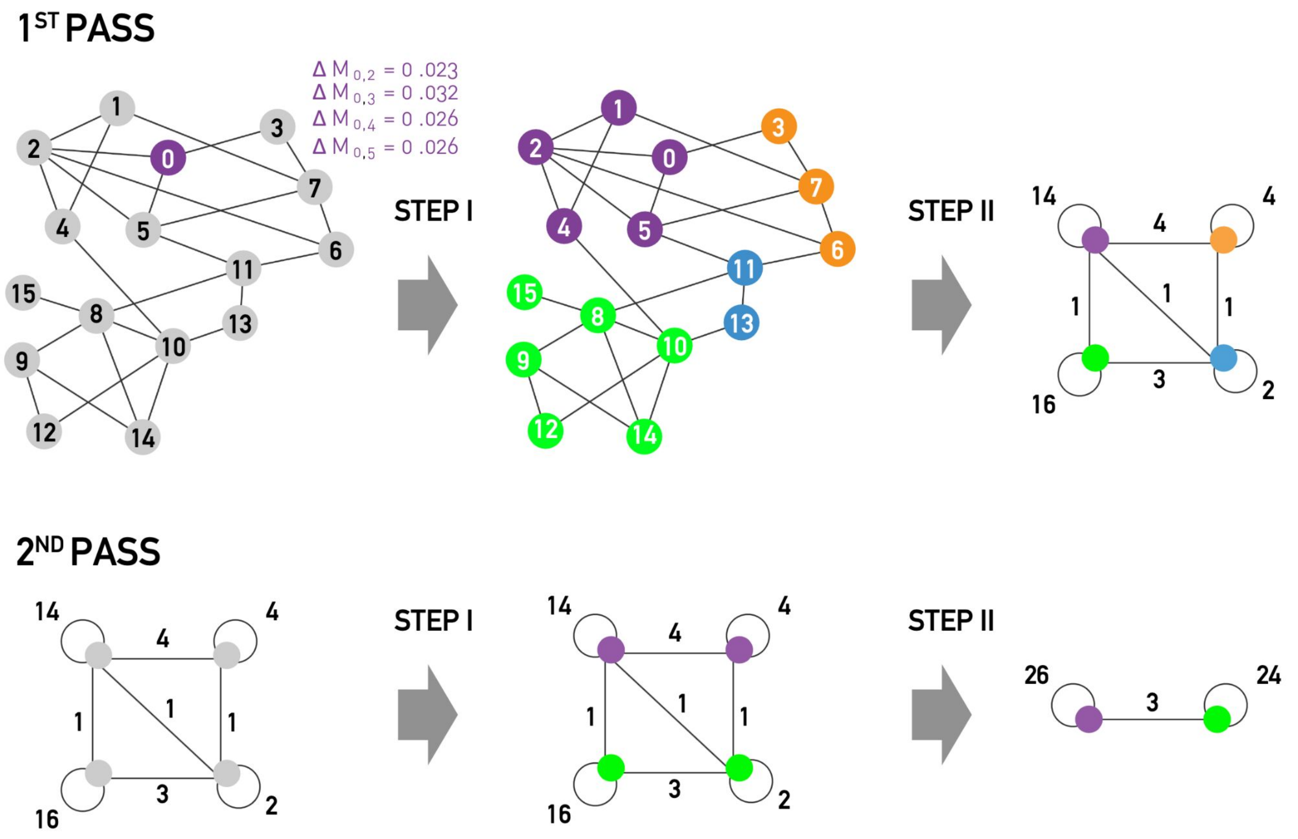
The Louvain algorithm



The sum of Steps I & II is called a pass.

The network obtained after each pass is processed again (Pass 2), until no further increase of modularity is possible.

The Louvain algorithm



Pass 1 is the most time consuming: The number of computations scale linearly with L.

With subsequent passes over a decreasing number of nodes and links, the complexity of the algorithm is at most O(L).

It therefore allows us to identify communities in networks with millions of nodes.

Still greedy. And result depends on the order in which the nodes are visited. But fast $\rightarrow$ very commonly used in practice.

More later

3.3. Label propagation

3.4. Stochastic block modeling

Summary

Communities play a key role in the structure and function of networks.

But communities are not well-defined objects.

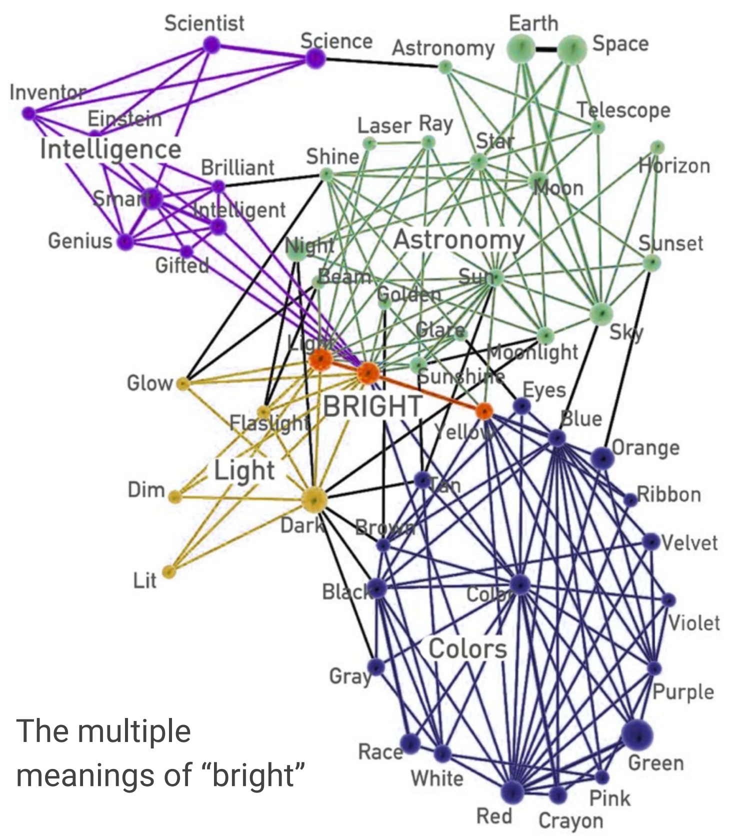
Network partitioning searches for well-separated subnetworks.

Hierarchical clustering groups nodes based on their similarity. Biggest drawback: lack of criterion for selecting meaningful partitions.

Bridge removal (same drawback).

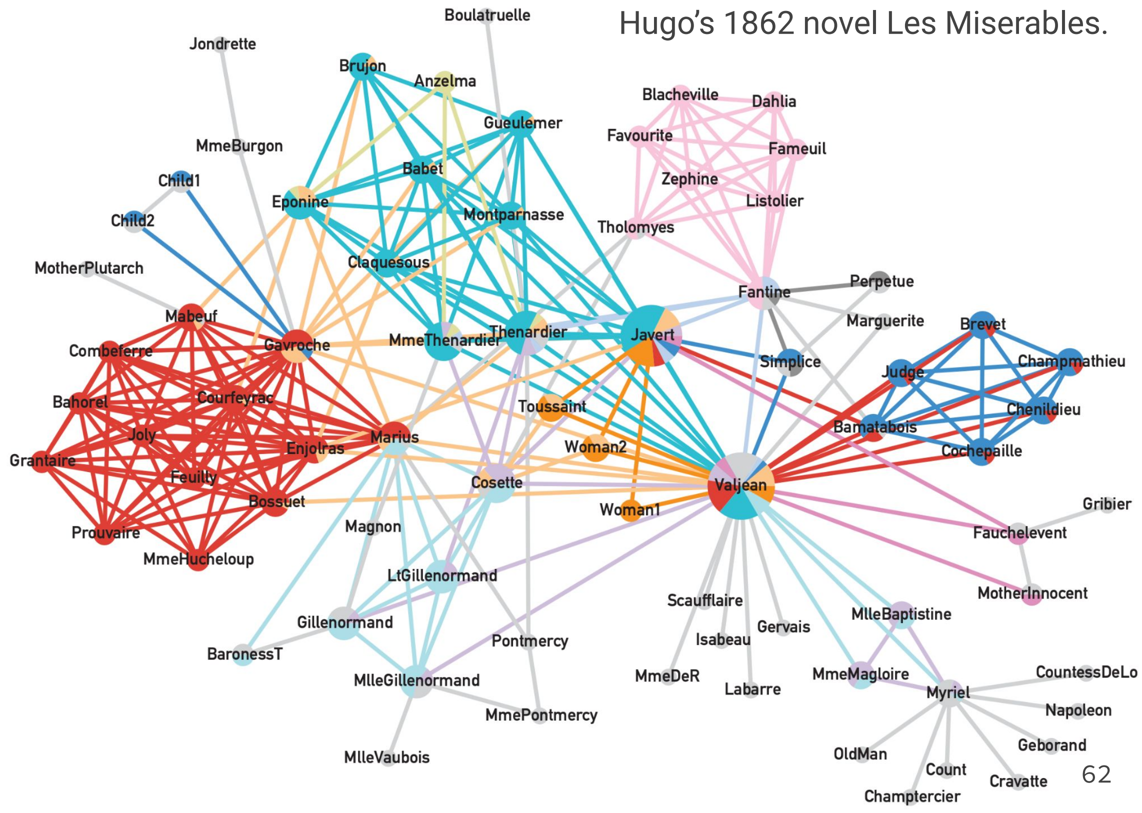
Modularity optimization (Louvain) widely used in practice.

Big limitation so far: A node rarely belongs to a single community!



The multiple meanings of “bright”

The network of characters in Victor Hugo’s 1862 novel Les Miserables.


Boulatruelle
Jondrette
Brujon
Anzelma
Blacheville
Dahlia
Gueulemer
Favourite
Fameuil
MmeBurgon
Babet
Zephine
Child1
Eponine
Listolier
Child2
Montparnasse
Tholomyes
Claquesous
MotherPlutarch
Fantine
Perpetue
Mabeuf
Thenardier
Marguerite
Brevet
MmeThenardier
Javert
Combeferre
Gavroche
Simplice
Champmathieu
Judge
Courfeyrac
Bahorel
Toussaint
Chenildieu
Bamatabois
Joly
Marius
Enjolras
Woman2
Cochepaille
Grantaire
Feuilly
Cosette
Valjean
Bossuet
Gribier
Woman1
Magnon
Prouvaire
Fauchelevent
MmeHucheloup
LtGillenormand
MotherInnocent
Scaufflaire
MlleBaptistine
Gervais
Gillenormand
Isabeau
Pontmercy
BaronessT
MmeDeR
MmeMagloire
MlleGillenormand
Labarre
CountessDeLo
Myriel
Napoleon
MmePontmercy
Geborand
OldMan
MlleVaubois
Count
Cravatte
Champtercier